Lecture 15. Nonhomogeneous Equations and Variation of Parameters

Nonhomogeneous Equations (Review and Generalization)

Recall we talked about the solution of general nonhomogeneous nth-order linear equation with constant coefficients in Lecture 14. More generally, we consider the following:

Nonhomogeneous Equations

Now we consider the *nonhomogeneous* nth-order linear differential equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = f(x) \tag{1}$$

with associated homogeneous equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0 \tag{2}$$

THEOREM 5 Solutions of Nonhomogeneous Equations

Let y_p be a particular solution of the nonhomogeneous equation in (1) on an open interval I where the functions p_i and f are continuous. Let $y_1, y_2, \cdots y_n$ be linearly independent solutions of the associated homogeneous equation in (2). If Y is any solution whatsoever of Eq. (1) on I, then there exist numbers $c_1, c_2, \cdots, c_n$ such that

$$Y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n + y_p(x) = y_c + y_p$$

for all x in I.

Recall in **Lecture 14**, we discussed the method of Undetermined Coefficients of finding a particular solution y_p. In this lecture, we discuss another method to find y_p.

Variation of Parameters

THEOREM 1 Variation of Parameters

If the nonhomogeneous equation $y'' + P(x)y' + Q(x)y = f(x)$ has complementary function $y_c(x) = c_1y_1(x) + c_2y_2(x)$, then a particular solution is given by

$$y_p(x) = -y_1(x)\int \frac{y_2(x)f(x)}{W(x)}dx + y_2(x)\int \frac{y_1(x)f(x)}{W(x)}dx$$

where $W = W(y_1, y_2)$ is the Wronskian of the two independent solutions y_1 and y_2 of the associated homogeneous equation.

Remark. A proof of this can be found in the book Section 3.6.

If we write $u_1(x) = -\int \frac{y_2(x)f(x)}{W(x)}dx$, $u_2(x) = \int \frac{y_1(x)f(x)}{W(x)}dx$

then $y_p(x) = u_1y + u_2y_2$

Example 1 Use the method of variation of parameters to find a particular solution of the given differential equation.

$$y'' + 9y = 2\sec 3x = \frac{2}{\cos 3x} = f(x)$$

ANS: We consider the corresponding homogeneous eqn to find two fundamental solutions y_1 and y_2.

Consider $y'' + 9y = 0$

The char. eqn is $r^2 + 9 = 0 \Rightarrow r^2 = -9 \Rightarrow r = \pm 3i$

Then $y_c(x) = c_1y_1 + c_2y_2 = c_1 \underbrace{\cos 3x}_{y_1} + c_2 \underbrace{\sin 3x}_{y_2}$

We have

$$W(x) = W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix}$$

$$= 3\cos^2 3x + 3\sin^2 3x = 3(\overbrace{\cos^2 3x + \sin^2 3x}^{1}) = 3$$

We have

$$u_1(x) = -\int \frac{y_2(x) f(x)}{W(x)}\,dx = -\int \frac{\sin 3x \cdot 2 \cdot \frac{1}{\cos 3x}}{3}\,dx$$

$$= -\frac{2}{3}\int \tan 3x\,dx = -\frac{2}{3}\,\frac{1}{3}\int \tan 3x\,d(3x)$$

$$= -\frac{2}{9}\int \tan 3x\,d(3x) = \frac{2}{9}\ln|\cos 3x|$$

Recall $\int \tan t\,dt = -\ln|\cos t| + C$

$$u_2(x) = \int \frac{y_1(x) f(x)}{W(x)}\,dx = \int \frac{\cos 3x \cdot 2 \cdot \frac{1}{\cos 3x}}{3}\,dx = \frac{2}{3}\int 1\,dx = \frac{2}{3}x$$

So $\boxed{y_p(x) = u_1 y_1 + u_2 y_2 = \frac{2}{9}\cos 3x \ln|\cos 3x| + \frac{2}{3}x\sin 3x}$

And the general sol is

$$y(x) = y_c + y_p$$

$$= C_1\cos 3x + C_2 \sin 3x + \frac{2}{9}\cos 3x \ln|\cos 3x| + \frac{2}{3}x\sin 3x$$

Example 2.

Find a particular solution to

$$y'' + 8y' + 16y = \frac{-8e^{-4t}}{t^2+1} = f(t)$$

ANS: First we want to find y_1 and y_2 for the homogenous part.

$$y'' + 8y' + 16y = 0$$

The char. eqn is $r^2 + 8r + 16 = 0$

$$\Rightarrow (r+4)^2 = 0 \Rightarrow r_{1,2} = -4$$

Thus $y_c(t) = c_1 y_1 + c_2 y_2 = c_1 \underbrace{e^{-4t}}_{y_1} + c_2 \underbrace{te^{-4t}}_{y_2}$

Compute the Wronskian of $y_1(t)$ and $y_2(t)$.

$$W(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-4t} & te^{-4t} \\ -4e^{-4t} & e^{-4t} - 4te^{-4t} \end{vmatrix} = e^{-4t}(e^{-4t} - 4te^{-4t}) + 4te^{-4t} \cdot e^{-4t}$$

$$= e^{-8t} - \cancel{4te^{-8t}} + \cancel{4te^{-8t}} = e^{-8t}$$

We know $f(t) = -\dfrac{8e^{-4t}}{t^2+1}$

Compute $u_1(t) = -\displaystyle\int \frac{y_2(t) f(t)}{W(t)}\, dt = +\int \frac{te^{-4t} \cdot \frac{8e^{-4t}}{t^2+1}}{e^{-8t}}\, dt$

$$= \int \frac{8t}{t^2+1}\, dt = 4\int \frac{d(t^2+1)}{t^2+1} = 4\ln(t^2+1)$$

$$u_2(t) = \int \frac{f(t)\, y_1(t)}{W(t)}\, dt = -\int \frac{\cancel{e^{-4t}} \cdot \frac{8\cancel{e^{-4t}}}{t^2+1}}{\cancel{e^{-8t}}}\, dt$$

$$= -\int \frac{8}{t^2+1}\, dt = -8 \tan^{-1}(t)$$

Thus, a particular solution is

$$y_p(t) = u_1(t) y_1(t) + u_2(t) y_2(t)$$

$$= 4e^{-4t} \ln(t^2+1) - 8e^{-4t}\, t \tan^{-1}(t)$$